$$= \frac{\partial^2 \cancel{F_3}}{\partial \cancel{x} \partial y} - \frac{\partial^2 \cancel{F_2}}{\cancel{\partial x} \partial z} + \frac{\partial^2 \cancel{F_1}}{\partial \cancel{y} \partial z} - \frac{\partial^2 \cancel{F_3}}{\cancel{\partial y} \partial x} + \frac{\partial^2 \cancel{F_2}}{\cancel{\partial z} \partial x} - \frac{\partial^2 \cancel{F_1}}{\cancel{\partial z} \partial y}$$

$$= \underline{0}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

5) If $\vec{F} = x^2 y \hat{i} - 2xz \hat{j} + 2yz \hat{k}$
find Curl(Curl $\vec{F}$) or $\nabla \times (\nabla \times \vec{F})$.

<u>solution:</u> $\text{Curl } \vec{F} = (2z + 2x)\hat{i} - (2z + x^2)\hat{k}$

$$\text{curl}(\text{curl}\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z+2x & 0 & -(2z+x^2) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-2z+x^2) - \frac{\partial}{\partial z}(0)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-2z+x^2) - \frac{\partial}{\partial z}(2z+2x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(2z+2x)\right]$$

$$= 0 - j\left[-2x - 2\right] + 0$$

$$= \underline{(2x+2)\hat{j}}$$

6) Show that $\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find ϕ such that $\vec{F} = \nabla\phi$

<u>Solution:</u> We have to show that $\nabla \times \vec{F} = 0$

$$\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right]$$

$$= \hat{i}\left[-1 + 1\right] - \hat{j}\left[3z^2 - 3z^2\right] + \hat{k}\left[6x - 6x\right]$$

$$= 0$$

$\therefore \vec{F}$ is irrotational.

To find ϕ such that $\vec{F} = \nabla\phi$

$$(6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 6xy + z^3 \quad \Bigg| \quad \frac{\partial\phi}{\partial y} = 3x^2 - z \quad \Bigg| \quad \frac{\partial\phi}{\partial z} = 3xz^2 - y$$

$\Rightarrow \cancel{\phi = \frac{6x^2}{2}y + xz^3}$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$d\phi = (6xy + z^3)dx + (3x^2 - z)dy + (3xz^2 - y)dz$$

$$d\phi = \underline{6xy\,dx} + \underline{z^3dx} + \underline{3x^2dy} - zdy + 3x\underline{z^2dz} - ydz$$

$$d\phi = (6xy\,dx + 3x^2dy) + (z^3dx + 3xz^2dz)$$

$$- (zdy + ydz)$$

$$d\phi = d(3x^2y) + d(z^3x) - d(yz)$$

$$\phi = 3x^2y + z^3x - yz + \underline{\underline{C}}$$

7) Find the constants a, b, c so that the vector field $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

<u>Solution</u>: Given $\vec{F}$ is irrotational, $\nabla \times \vec{F} = 0$

$$\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\hat{i}(c+1) + \hat{j}[4-a] + \hat{k}(b+2) = 0$$

$$(c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k} = 0$$

$$\Rightarrow c+1=0, \quad a-4=0, \quad b-2=0$$

$$\Rightarrow c=-1, \quad a=4, \quad b=2$$

<u>H.W.</u> If $\vec{F} = (x+y+az)\hat{i} + (bx+2y-z)\hat{j} + (x+cy+2z)\hat{k}$ find a, b, c such that $\vec{F}$ is irrotational then find ϕ such that $\vec{F} = \nabla\phi$.

Q) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that $r^n\vec{r}$ is irrotational for all values of n and solenoidal only for $n=-3$.

<u>Solution:</u> $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

We have $\text{curl}\, \phi\vec{F} = \phi\, \text{curl}\, \vec{F} + \nabla\phi \times \vec{F}$

$\vec{F} = \vec{r}, \quad \phi = r^n$

$$\text{curl}\, r^n\vec{r} = r^n\, \text{curl}\, \vec{r} + \nabla r^n \times \vec{r} \rightarrow (1)$$

$$\text{curl}\, \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$= \hat{i}\,[0-0] - \hat{j}\,[0-0] + \hat{k}\,[0-0]$

$= \vec{0} \rightarrow (2)$

Sub (2) in (1)

$$\text{curl}(r^n \vec{r}) = r^n \cdot \vec{0} + n r^{n-1} \frac{\vec{r}}{r} \times \vec{r}$$

$$= n r^{n-2} (\vec{r} \times \vec{r}) \rightarrow (3)$$

$$\vec{r} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ x & y & z \end{vmatrix}$$

$= \hat{i}\,[yz - yz] - \hat{j}\,[xz - xz] + \hat{k}\,[xy - xy]$

$= 0 \rightarrow (4)$ Sub (4) in (3)

$$\text{curl}(r^n \vec{r}) = n r^{n-2} (0)$$

$$= 0$$

$\therefore$ Curl$(r^n \vec{r})$ is irrotational for all n.

$$\text{div}(r^n \vec{r}) = (n+3) r^n$$

put $n = -3$

$$\text{div}(r^{-3} \vec{r}) = (-3+3) r^{-3} = 0$$

$\therefore$ $r^n \vec{r}$ is solenoidal only if $n = -3$

10) Show that $\vec{F} = (2xy^2+yz)\hat{i} + (2x^2y+xz+2yz^2)\hat{j} + (2y^2z+xy)\hat{k}$ is a Conservative force field and find its scalar potential.

Solution: We know that every Conservative vector is an irrotational vector.

we need to show that curl $\vec{F} = 0$

$$\nabla \times \vec{F} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^2+yz & 2x^2y+xz+2yz^2 & 2y^2z+xy \end{vmatrix}$$

$$= \hat{i}\left[(4yz+x) - (4yz+x)\right] - \hat{j}\left[y-y\right] + \hat{k}\left[(4xy+z)-(4xy+z)\right]$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= 0 //$$

To find ϕ, $\vec{F} = \nabla\phi$

$$(2xy^2+yz)\hat{i} + (2x^2y+xz+2yz^2)\hat{j} + (2y^2z+xy)\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 2xy^2 + yz \,; \quad \frac{\partial \phi}{\partial y} = 2x^2y + xz + 2yz^2 \,;$$

$$\frac{\partial \phi}{\partial z} = 2y^2z + xy$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$= (2xy^2 + yz)\,dx + (2x^2y + xz + 2yz^2)\,dy$$

$$+ \left[(2y^2z) + xy\right] dz$$

$$= \underline{2xy^2dx} + yzdx + \underline{2x^2y\,dy} + xz\,dy + \underline{2yz^2dy}$$

$$+ \underline{2y^2z\,dz} + xydz$$

$$= (2xy^2dx + 2x^2ydy) + (2yz^2dy + 2y^2z\,dz)$$

$$+ (yz\,dx + xz\,dy + xy\,dz)$$

$$d\phi = d(x^2y^2) + d(y^2z^2) + d(xyz)$$

$$\phi = x^2y^2 + y^2z^2 + xyz$$

$\therefore$ ϕ is scalar potential